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Heat transfer in a rarefied polyatomic gas—I. Plane parallel plates

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Abstract—The Hanson–Morse model for a polyatomic gas is used to investigate heat transfer between two plane parallel plates. The boundary value problem is transformed into a system of integral equations which is then solved numerically. The dependence of heat transfer, temperature and density profiles on Knudsen number, internal energy, Eucken factor and collision relaxation number is studied. Results are compared with the available experimental data on nitrogen and some previously reported variational results on heat transfer.

1. INTRODUCTION

THERE HAS been considerable interest in the problem of heat conduction through rarefied gases. Although the kinetic theory of plane heat conduction in monatomic gases has been investigated extensively [1–5], the corresponding problem for polyatomic gases has received less attention. The available work includes the variational results on parallel plate heat transfer by Hsu and Morse [6–8] and Cipolla [9], and experimental work by Teagan and Springer [10].

Hsu and Morse used the Hanson–Morse kinetic model equation [11] with a boundary condition expressed in terms of translational and internal thermal accommodation coefficients. They converted the problem to an integral equation and obtained heat transfer rates by use of a variational technique. The authors reported good agreement with the experimental data of Teagan and Springer. Since Teagan and Springer had also reported results for density profiles, Hsu and Morse attempted to make some comparisons for these also by using estimates based on trial functions of the variational approach. These comparisons showed disagreement, which is not surprising as the variational technique is not suitable for providing estimates of density profiles.

Cipolla used a variational technique based on the integro-differential form of the linearized Wang Chang–Uhlenbeck equation. He obtained an algebraic expression for the normalized heat transfer between parallel plates. This expression has correct limiting forms (continuum and free molecule) and the heat transfer rate can be computed easily for arbitrary Knudsen number.

The experimental results on heat transfer and density distribution measurements between parallel plates are due to Teagan and Springer [10], who used argon and nitrogen as test gases. The density profile of each gas was determined by measuring the luminescence produced by the passage of a narrow, constant current beam of high energy electrons through the test gases. The heat transfer between the plates was determined by measuring the total electrical power input into the hot plate heating element after steady-state condition had been reached. The thermal accommodation coefficients were determined from heat transfer measurements made at low pressure (free molecule condition), with the assumption of equal and uniform accommodation coefficients for hot and cold plates.

In this paper, the kinetic boundary value problem of heat transfer between parallel plates is considered. We use the Hanson–Morse kinetic model equation, as given by Hsu and Morse [7] and Cipolla [9]. We convert the relevant integro-differential equation with the associated boundary conditions into a system of integral equations, which is then solved by a straightforward numerical technique [12–15]. This enables us to verify the accuracy of the variational results, and also provides accurate results for density and temperature distributions.

Our results are in good agreement with those of Hsu and Morse. Good agreement with experimental values of Teagan and Springer is also obtained provided certain choices of parameters are made. The present results are generally in good agreement with Cipolla's variational results, but for some combinations of parameters certain disagreements are noted.

2. FORMULATION OF THE PROBLEM

2.1. *Wang Chang–Uhlenbeck equation*

For polyatomic gases the molecules are treated semi-quantum mechanically in the sense that the internal energy of the particles is quantized, but the translational energy is treated classically. To each internal quantum state with internal energy E_i, is assigned a velocity distribution function f_i. The spatial and temporal evolution of f_i due to collision as given by Wang Chang *et al.* [16] is described by:

$$\frac{\partial}{\partial \tilde{t}} f_i + \tilde{\boldsymbol{\xi}} \cdot \frac{\partial}{\partial \tilde{\mathbf{x}}} f_i = \sum_{jkm} \int [f'_k f'_{ms} - f_i f_{js}] \tilde{g} \tilde{I}_{ij}^{km}(\tilde{g}, \theta) \, \mathrm{d}\omega \, \mathrm{d}\tilde{\boldsymbol{\xi}}_s \quad (1)$$

NOMENCLATURE

$\mathbf{c}$	dimensionless molecular velocity, $\tilde{\boldsymbol{\xi}}(2RT_0)^{-1/2}$
c_v^i	internal specific heat per molecule
c_v	total specific heat, $(3/2)k + c_v^i$
E_i	internal energy of level i (dimensional)
f_i	molecular velocity distribution for particles in level i
f_{0i}	absolute Maxwell–Boltzmann distribution
F_t	total Eucken number
$\tilde{g}$	dimensional relative velocity
G	dimensionless internal specific heat, c_v^i/k
h	perturbation of distribution
$\tilde{I}_{ij}^{km}$	dimensional scattering cross-section
k	Boltzmann constant
l	mean free path
m	molecular mass
n_0	number density at $x = 0$
q	total heat flux in x direction
q_{fm}	free molecule heat flux in x direction
q_{tr}	translational heat flux in x direction
q_{int}	internal heat flux in x direction
Q	partition function, $\sum \exp(-E_s/kT)$
R	gas constant
T_w, T_0	wall temperature and temperature at $x = 0$, respectively
T_t	total temperature
T_{tr}, T_{int}	translational and internal temperature, respectively
$\bar{T}_{tr}$, $\bar{T}_{int}$	perturbed translational and internal temperature, respectively
$\tilde{\mathbf{x}}$	spatial coordinate
x	dimensionless coordinate perpendicular to the plates
Z	collision relaxation number.

Greek symbols

α_{tr}, α_{int}	thermal accommodation coefficients for translational and internal energy, respectively
δ	inverse Knudsen number
ε_i	dimensionless internal energy
θ	collisional scattering angle
$\tilde{\boldsymbol{\xi}}$	molecular velocity (dimensional)
ρ	gas density
$\bar{\rho}$	perturbed gas density
τ	temperature perturbation, $\lvert(T_w - T)/T_0\rvert$
τ_r	relaxation time for the internal degree of freedom
τ_c	inverse of the total collision frequency
ω	solid angle of scattering.

where tildes denote dimensional quantities, and in standard notation: $\tilde{g}$ is the relative velocity before collision, $\tilde{I}_{ij}^{km}$ is the collision cross section, and f'_{ms} denotes $f_m(\tilde{\boldsymbol{\xi}}'_s)$ where primes indicate post-collisional quantities.

The steady-state form of equation (1) can be linearized by considering f_i to be a small deviation from the Maxwell–Boltzmann distribution f_{0i} such that:

$$f_i = f_{0i}(1 + h_i), \quad \text{and} \quad |h_i| \ll 1 \tag{2}$$

where

$$f_{0i} = n_0(2\pi RT_0)^{-3/2} Q_0^{-1} \exp(-c^2 - \varepsilon_i) \tag{3}$$

and

$$\varepsilon_i = \frac{E_i}{kT_0}, \quad Q_0 = \sum \exp(-\varepsilon_s), \quad c^2 = \frac{\xi^2}{2RT_0}.$$

Here h is a measure of the perturbation on the distribution function from the local Maxwellian, n_0 is the number density and R is the gas constant.

Substitution of equation (2) into the steady-state form of (1) results in the following:

$$\tilde{\boldsymbol{\xi}} \cdot \frac{\partial h}{\partial \tilde{\mathbf{x}}} = n_0 Lh \tag{4}$$

$$Lh = \sum_{jkm} \int \frac{\exp(-c_s^2 - \varepsilon_j)}{Q_0 \pi^{3/2}} \times [h'_k + h'_{ms} - h_i - h_{js}] \tilde{g} \tilde{I}_{ij}^{km} \, d\omega \, d\mathbf{c}_s. \tag{5}$$

2.2. *Kinetic model and the associated boundary conditions*

Consider a polyatomic gas confined between two parallel, stationary plates in the planes $x = \pm\delta/2$ with the temperature $T_w\pm = T_0(1 \mp \tau)$, where $\tau = \Delta T/T_0 \ll 1$. Neglecting variations in the directions parallel to the plates leads to the one-dimensional equation:

$$c_x \frac{\partial h}{\partial x} = Lh \quad x \in (-\delta/2, \delta/2) \tag{6}$$

$$h^+ = h_0 + Ah^- \quad x = -\delta/2 \operatorname{sgn} c_x \tag{7}$$

where distance is measured in units of a mean free path l, and $\delta = d/l$ is an inverse Knudsen number. In the collision term L, $\tilde{g}\tilde{I}_{ij}^{km}$ is nondimensionalized by $n_0(2RT_0)^{-1/2}$. The constants n_0 and T_0 are chosen to be the density and temperature of the gas at $x = 0$. We will assume that the perturbed distribution emitted from

the wall may be written as [9]:

$$h_i^+ = \gamma + \tau_{tr}(c^2-2) + \tau_{int}(\varepsilon_i - G)$$
$$x = -\delta/2 \text{ sgn } c_x \quad (8)$$

where γ, τ_{tr}, τ_{int} are expressed as:

$$\gamma = (1, h^-) \quad (9)$$

$$\tau_{tr} = \tau\alpha_{tr} + [(1-\alpha_{tr})/2](c'^2 - 2, h^-) \quad (10)$$

$$\tau_{int} = \tau\alpha_{int} + (1-\alpha_{tr})(1/G)(\varepsilon_j - G, h^-) \quad (11)$$

where

$$(f, h^-) = \sum_j \frac{2}{\pi} \int_{\mathbf{c}'\cdot\hat{n}<0} d\mathbf{c}' \times \frac{\exp(-c'^2-\varepsilon_j)}{Q_0} |\mathbf{c}'\cdot\hat{n}| f h^-(-\delta/2, \mathbf{c}', \varepsilon_j) \quad (12)$$

and $G = c_v^i/k = Q_0^{-1}\sum \varepsilon_i \exp(-\varepsilon_i)$. The thermal accommodation coefficients for the internal and translational energy are, respectively, α_{int} and α_{tr}.

Following Hanson and Morse [6–9], the operator L is approximated by:

$$L(h) = -h(x, \mathbf{c}, \varepsilon_i) + \sum_{m=1}^{5} \psi_m(\mathbf{c}, \varepsilon_i) a_m(x) \quad (13)$$

where

$$\psi_1 = 1 \quad (14)$$

$$\psi_2 = (2/3)[c^2 - (3/2)][1 - (2G/3Z)] + (2/3Z)(\varepsilon_i - G) \quad (15)$$

$$\psi_3 = (2/3)[c^2 - (3/2)](1/Z) + (1/G)(\varepsilon_i - G)[1 - (1/Z)] \quad (16)$$

$$\psi_4 = (4/9)c_x[c^2 - (5/2)][1 - (G/Z)] + c_x(\varepsilon_i - G)(2/3Z) \quad (17)$$

$$\psi_5 = c_x[c^2 - (5/2)][2/(3Z)] + c_x(\varepsilon_i - G)(1-F)(2/G). \quad (18)$$

The moments $a_m(x)$ are expressed as:

$$a_m(x) = [(\varphi_m, h_i)] = \sum_i \frac{1}{\pi^{3/2}} \times \int d\mathbf{c} \frac{\exp(-c^2-\varepsilon_i)}{Q_0} h_i(x, \mathbf{c}, \varepsilon_i)\varphi_m(\mathbf{c}, \varepsilon_i) \quad (19)$$

and they are related to macroscopic quantities of interest:

$$a_1(x) = \bar{\rho} = [\rho(x) - \rho(0)]/\rho(0)$$
$$a_2(x) = \bar{T}_{tr} = [T_{tr}(x) - T_{tr}(0)]/T_{tr}(0)$$
$$a_3(x) = \bar{T}_{int} = [T_{int}(x) - T_{int}(0)]/T_{int}(0)$$
$$a_4(x) = q_{tr}$$
$$a_5(x) = q_{int}$$

where ρ, T_{tr} and T_{int} represent density, translational and internal temperature, respectively. Also q_{tr} and q_{int} are, respectively, the translational and internal heat flux in the x direction.

The functions φ_m are defined as follows:

$$\varphi_1(\mathbf{c}, \varepsilon_i) = 1 \quad (20)$$
$$\varphi_2(\mathbf{c}, \varepsilon_i) = c^2 - 3/2 \quad (21)$$
$$\varphi_3(\mathbf{c}, \varepsilon_i) = \varepsilon_i - G \quad (22)$$
$$\varphi_4(\mathbf{c}, \varepsilon_i) = c_x[c^2 - (5/2)] \quad (23)$$
$$\varphi_5(\mathbf{c}, \varepsilon_i) = c_x(\varepsilon_i - G). \quad (24)$$

The total heat flux is expressed as:

$$q = q_{tr} + q_{int}. \quad (25)$$

Further, the total temperature is defined such that

$$c_v T_t = (3/2)kT_{tr} + c_v^i T_{int} \quad (26)$$

where $c_v = (3/2)k + c_v^i$ represents the total constant volume specific heat of the gas.

There are three dimensionless parameters in the model operator: (1) $G = c_v^i/k$, where c_v^i is the internal specific heat; (2) Z, the collisional relaxation number; and (3) F_t, the total Eucken number. The collision relaxation number has been defined as:

$$Z = \tau_r/\tau_c \quad (27)$$

where τ_r and τ_c are experimental and collision relaxation times. The quantity F is given by:

$$F = \frac{(10/9)(G/Z) + (2G/3)(4/9 + 5G/9Z) + (5G/18Z^2)(c_v/k)F_t}{(4/9 + 5G/9Z)(c_v/k)F_t - (5/3)}. \quad (28)$$

The ratio of heat transfer along the x axis to the corresponding value in free molecular condition is given by:

$$\frac{q}{q_{fm}} = \frac{\pi^{1/2}}{2\tau b} q \quad (29)$$

where q is given by equation (25) and the quantity b is:

$$b = \frac{\alpha_{tr}}{2-\alpha_{tr}} + \frac{\alpha_{int}}{2-\alpha_{int}}. \quad (30)$$

3. METHOD OF SOLUTION

Integration of equation (6) with the boundary condition given by (8) gives the following expression for h_i:

$$h(x, \mathbf{c}, \varepsilon_i) = \eta(c_x)\{[\gamma + \tau_{tr}(c^2-2) + \tau_{int}(\varepsilon_i - G)] \times \exp[-(\delta/2 + x)/c_x]\} - \eta(-c_x)\{[\gamma + \tau_{tr}(c^2-2) + \tau_{int}(\varepsilon_i - G)] \exp[-(\delta/2 - x)/c_x]\} + 1/c_x \Big\{ \eta(c_x) \int_{-\delta/2}^{x} \exp[(x'-x)/c_x] \times \sum_{m=1}^{5} \psi_m(\mathbf{c}, \varepsilon_i) a_m(x')\, dx' - \eta(-c_x) \times \int_{x}^{\delta/2} \exp[(x-x')/c_x] \sum_{m=1}^{5} \psi_m(\mathbf{c}, \varepsilon_i) a_m(x')\, dx' \Big\} \quad (31)$$

where

$$\eta(x) = 1 \quad x \geqslant 0$$

$$\eta(x) = 0 \quad x < 0.$$

Incorporation of the above expression into equation (19) results in the following system of integral equations for a_m:

$$a_m(x) = \gamma S_m(x) + \tau_{tr} J_m(x) + \tau_{int} L_m(x) + \sum_{j=1}^{5} \int K_{mj}(x, x') a_j(x')\, dx' \quad (32)$$

where $S_m(x)$, $J_m(x)$, $L_m(x)$ and $K_{mj}(x, x')$ are given in the Appendix. The functions $a_m(x)$ can be written as:

$$a_m(x) = \gamma P_m(x) + \tau_{tr} Q_m(x) + \tau_{int} R_m(x) \quad (33)$$

where

$$P_m(x) = S_m(x) + \int_{-\delta/2}^{\delta/2} \sum_{j=1}^{5} K_{mj}(x, x') P_j(x')\, dx' \quad (34)$$

$$Q_m(x) = J_m(x) + \int_{-\delta/2}^{\delta/2} \sum_{j=1}^{5} K_{mj}(x, x') Q_j(x')\, dx' \quad (35)$$

$$R_m(x) = L_m(x) + \int_{-\delta/2}^{\delta/2} \sum_{j=1}^{5} K_{mj}(x, x') R_j(x')\, dx'. \quad (36)$$

Using equations (9)–(11), γ, τ_{tr} and τ_{int} can be expressed as:

$$\begin{bmatrix} \gamma \\ \tau_{tr} \\ \tau_{int} \end{bmatrix} = \tau \mathbf{C}^{-1} \begin{bmatrix} 0 \\ \alpha_{tr} \\ \alpha_{int} \end{bmatrix} \quad (37)$$

where elements of matrix **C** are given in the Appendix.

Integral equations (34)–(36) were solved using a 41-point Gauss–Kronrod quadrature and the numerical method given in refs. [12–15]. The 81-point Gauss–Kronrod quadrature was tried once but was avoided on later calculations due to high computer cost and insignificant changes in the results. Once $P_m(x)$, $Q_m(x)$ and $R_m(x)$ are known, γ, τ_{tr} and τ_{int} are obtained immediately from equation (37).

4. NUMERICAL RESULTS AND DISCUSSION

We have calculated heat transfer ratio, temperature (internal, translational and total) and density profiles for N_2, air, CO_2 and SO_2 with physical parameters as given by Teagan and Springer and at temperatures of 543, 645 and 645 K, respectively. Tables 1–4 list values for the heat transfer ratio for CO_2, air, SO_2 and N_2, respectively, for all combinations of translational and internal accommodation coefficients of 0.2, 0.6 and 1.0. Table 5 lists density and temperature (translational, internal and total) values of N_2 for inverse Knudsen number of 2. All numerical values are truncated at the fourth decimal place. The detailed results for N_2, for which experimental data are available, are discussed below.

In our numerical calculations, we have varied δ between 0.01 and 40. The present results for the ratio of heat transfer to the corresponding free-molecular conditions are compared with the variational results of Hsu and Morse [7]. The results are also compared with the experimental results for N_2 assuming values of the accommodation coefficients as given by Teagan and Springer [10] (Fig. 1). The experimental data of Teagan and Springer, which correspond to the transition region, are in excellent agreement with ours for small values of δ but the differences become larger as δ

Table 1. Heat transfer ratio q/q_{fm} vs inverse Knudsen number ($1/Kn$) for CO_2: $F_t = 1.629$, $Z = 7$, $G = 1.412$

$1/Kn$	$\alpha_{tr} = 0.2$ ($\alpha_{int} = 0.2, 0.6, 1.0$)			$\alpha_{tr} = 0.60$ ($\alpha_{int} = 0.2, 0.6, 1.0$)			$\alpha_{tr} = 1.0$ ($\alpha_{int} = 0.2, 0.6, 1.0$)		
0.01	0.9994	0.9977	0.9942	0.9981	0.9976	0.9948	0.9948	0.9957	0.9944
0.10	0.9945	0.9798	0.9509	0.9831	0.9793	0.9563	0.9558	0.9634	0.9529
0.50	0.9781	0.9227	0.8266	0.9351	0.9206	0.8444	0.8436	0.8682	0.8326
1.00	0.9615	0.8709	0.7303	0.8918	0.8666	0.7548	0.7578	0.7901	0.7359
1.25	0.9540	0.8493	0.6940	0.8738	0.8436	0.7199	0.7259	0.7592	0.6981
1.50	0.9468	0.8296	0.6629	0.8574	0.8223	0.6893	0.6987	0.7317	0.6650
1.75	0.9398	0.8114	0.6356	0.8424	0.8025	0.6621	0.6751	0.7070	0.6355
2.00	0.9331	0.7947	0.6116	0.8285	0.7840	0.6375	0.6543	0.6845	0.6089
2.50	0.9201	0.7643	0.5706	0.8034	0.7499	0.5945	0.6191	0.6446	0.5626
3.00	0.9076	0.7374	0.5368	0.7810	0.7191	0.5580	0.5900	0.6100	0.5234
4.00	0.8839	0.6911	0.4832	0.7421	0.6651	0.4984	0.5439	0.5522	0.4602
5.00	0.8614	0.6518	0.4417	0.7084	0.6190	0.4511	0.5076	0.5052	0.4109
7.00	0.8197	0.5872	0.3797	0.6513	0.5436	0.3798	0.4513	0.4320	0.3384
10.00	0.7635	0.5119	0.3150	0.5816	0.4589	0.3069	0.3889	0.3545	0.2671
15.00	0.6828	0.4203	0.2446	0.4920	0.3621	0.2311	0.3153	0.2712	0.1963
20.00	0.6145	0.3541	0.1984	0.4236	0.2967	0.1838	0.2633	0.2177	0.1537
25.00	0.5558	0.3037	0.1654	0.3695	0.2493	0.1511	0.2243	0.1803	0.1251
30.00	0.5047	0.2639	0.1407	0.3254	0.2132	0.1272	0.1939	0.1525	0.1045
35.00	0.4597	0.2316	0.1214	0.2888	0.1847	0.1089	0.1694	0.1310	0.0890
40.00	0.4197	0.2049	0.1059	0.2578	0.1617	0.0944	0.1493	0.1139	0.0768

Table 2. Heat transfer ratio q/q_{fm} vs inverse Knudsen number ($1/Kn$) for air: $F_t = 1.9843$, $Z = 25$, $G = 1.1799$

$1/Kn$	$\alpha_{tr} = 0.2$ ($\alpha_{int} = 0.2, 0.6, 1.0$)			$\alpha_{tr} = 0.60$ ($\alpha_{int} = 0.2, 0.6, 1.0$)			$\alpha_{tr} = 1.0$ ($\alpha_{int} = 0.2, 0.6, 1.0$)		
0.01	0.9994	0.9981	0.9950	0.9980	0.9977	0.9957	0.9951	0.9954	0.9947
0.10	0.9948	0.9834	0.9575	0.9828	0.9803	0.9638	0.9590	0.9614	0.9552
0.50	0.9794	0.9363	0.8491	0.9349	0.9251	0.8697	0.8548	0.8623	0.8412
1.00	0.9640	0.8927	0.7624	0.8910	0.8745	0.7922	0.7710	0.7811	0.7494
1.25	0.9571	0.8741	0.7288	0.8723	0.8530	0.7613	0.7381	0.7488	0.7135
1.50	0.9504	0.8570	0.6994	0.8551	0.8331	0.7340	0.7092	0.7202	0.6818
1.75	0.9440	0.8410	0.6734	0.8391	0.8146	0.7093	0.6833	0.6944	0.6536
2.00	0.9378	0.8261	0.6501	0.8240	0.7972	0.6869	0.6599	0.6709	0.6281
2.50	0.9259	0.7987	0.6098	0.7961	0.7652	0.6473	0.6189	0.6294	0.5834
3.00	0.9145	0.7740	0.5761	0.7708	0.7362	0.6131	0.5839	0.5936	0.5454
4.00	0.8927	0.7311	0.5222	0.7260	0.6852	0.5564	0.5265	0.5340	0.4834
5.00	0.8721	0.6945	0.4807	0.6870	0.6411	0.5106	0.4810	0.4860	0.4346
7.00	0.8337	0.6350	0.4204	0.6216	0.5684	0.4400	0.4123	0.4123	0.3618
10.00	0.7816	0.5673	0.3604	0.5448	0.4852	0.3655	0.3416	0.3357	0.2887
15.00	0.7057	0.4854	0.2968	0.4513	0.3882	0.2846	0.2663	0.2549	0.2146
20.00	0.6407	0.4241	0.2533	0.3832	0.3211	0.2315	0.2174	0.2038	0.1693
25.00	0.5839	0.3749	0.2202	0.3309	0.2718	0.1937	0.1823	0.1683	0.1386
30.00	0.5338	0.3341	0.1936	0.2892	0.2338	0.1651	0.1558	0.1422	0.1163
35.00	0.4893	0.2995	0.1716	0.2551	0.2036	0.1428	0.1350	0.1221	0.0993
40.00	0.4494	0.2698	0.1531	0.2266	0.1790	0.1248	0.1182	0.1061	0.0859

increases. It should be noted that an extensive comparison between our results and experimental values is not possible due to insufficient experimental data, experimental errors and, in particular, inaccuracy involved in evaluation of accommodation coefficients which could significantly influence the comparison between theory and experiment.

We have, however, used Cipolla's [9] variational formulation for heat transfer through a polyatomic gas between parallel plates to obtain numerical results for comparison purposes. His variational results are in good agreement with ours, but for certain combinations of accommodation coefficients the variational results oscillate beyond reasonable values. The reasons for this are under investigation.

The results for N_2 density distribution obtained by the present method and experimental values for values of $\delta = 2.58, 5.62, 11.25$ and 15.5 as given by Teagan and Springer are shown in Fig. 2. It is interesting to note that the density profile obtained by the present method tends to fit a straight line up to a certain distance from the center line and deviates from this line as it approaches the plate. The point of deviation gets farther away from the center line and closer to the plate

Table 3. Heat transfer ratio q/q_{fm} vs inverse Knudsen number ($1/Kn$) for SO_2: $F_t = 1.5838$, $Z = 25$, $G = 3.6124$

$1/Kn$	$\alpha_{tr} = 0.2$ ($\alpha_{int} = 0.2, 0.6, 1.0$)			$\alpha_{tr} = 0.60$ ($\alpha_{int} = 0.2, 0.6, 1.0$)			$\alpha_{tr} = 1.0$ ($\alpha_{int} = 0.2, 0.6, 1.0$)		
0.01	0.9994	0.9976	0.9942	0.9982	0.9976	0.9947	0.9954	0.9959	0.9943
0.10	0.9944	0.9794	0.9508	0.9848	0.9788	0.9549	0.9614	0.9654	0.9518
0.50	0.9775	0.9206	0.8248	0.9417	0.9185	0.8387	0.8628	0.8744	0.8287
1.00	0.9603	0.8663	0.7251	0.9019	0.8630	0.7452	0.7837	0.7979	0.7300
1.25	0.9525	0.8432	0.6868	0.8848	0.8393	0.7088	0.7529	0.7671	0.6917
1.50	0.9451	0.8221	0.6537	0.8691	0.8175	0.6769	0.7259	0.7395	0.6581
1.75	0.9379	0.8025	0.6244	0.8545	0.7973	0.6486	0.7018	0.7146	0.6282
2.00	0.9309	0.7843	0.5984	0.8407	0.7783	0.6230	0.6801	0.6917	0.6014
2.50	0.9174	0.7511	0.5538	0.8153	0.7435	0.5788	0.6423	0.6510	0.5548
3.00	0.9045	0.7216	0.5168	0.7921	0.7121	0.5413	0.6103	0.6156	0.5156
4.00	0.8800	0.6709	0.4585	0.7511	0.6575	0.4809	0.5581	0.5562	0.4524
5.00	0.8570	0.6286	0.4144	0.7153	0.6110	0.4337	0.5168	0.5079	0.4034
7.00	0.8144	0.5609	0.3510	0.6548	0.5354	0.3637	0.4540	0.4330	0.3317
10.00	0.7573	0.4859	0.2891	0.5825	0.4511	0.2935	0.3877	0.3542	0.2615
15.00	0.6759	0.3984	0.2251	0.4914	0.3554	0.2212	0.3133	0.2702	0.1919
20.00	0.6075	0.3362	0.1837	0.4227	0.2909	0.1762	0.2618	0.2166	0.1502
25.00	0.5488	0.2888	0.1540	0.3685	0.2443	0.1451	0.2233	0.1791	0.1223
30.00	0.4978	0.2513	0.1315	0.3245	0.2088	0.1223	0.1932	0.1514	0.1022
35.00	0.4530	0.2209	0.1138	0.2879	0.1809	0.1047	0.1690	0.1300	0.0869
40.00	0.4133	0.1955	0.0995	0.2570	0.1583	0.0908	0.1491	0.1130	0.0750

Table 4. Heat transfer ratio q/q_{fm} vs inverse Knudsen number ($1/Kn$) for N_2: $F_t = 1.96$, $Z = 5.08$, $G = 1.0$

$1/Kn$	$\alpha_{tr} = 0.2$ ($\alpha_{int} = 0.2, 0.6, 1.0$)			$\alpha_{tr} = 0.60$ ($\alpha_{int} = 0.2, 0.6, 1.0$)			$\alpha_{tr} = 1.0$ ($\alpha_{int} = 0.2, 0.6, 1.0$)		
0.01	0.9994	0.9981	0.9947	0.9979	0.9977	0.9958	0.9948	0.9952	0.9947
0.10	0.9948	0.9832	0.9553	0.9817	0.9801	0.9640	0.9560	0.9595	0.9549
0.50	0.9792	0.9353	0.8421	0.9303	0.9244	0.8708	0.8441	0.8557	0.8398
1.00	0.9636	0.8919	0.7552	0.8837	0.8732	0.7946	0.7558	0.7715	0.7472
1.25	0.9566	0.8738	0.7227	0.8640	0.8515	0.7646	0.7218	0.7384	0.7109
1.50	0.9499	0.8574	0.6949	0.8460	0.8314	0.7380	0.6923	0.7091	0.6790
1.75	0.9434	0.8423	0.6709	0.8293	0.8127	0.7140	0.6662	0.6829	0.6505
2.00	0.9371	0.8284	0.6497	0.8138	0.7951	0.6923	0.6429	0.6591	0.6248
2.50	0.9250	0.8034	0.6141	0.7854	0.7628	0.6540	0.6026	0.6172	0.5799
3.00	0.9134	0.7813	0.5850	0.7599	0.7335	0.6208	0.5686	0.5811	0.5416
4.00	0.8914	0.7435	0.5398	0.7151	0.6819	0.5656	0.5138	0.5216	0.4794
5.00	0.8706	0.7116	0.5053	0.6765	0.6376	0.5207	0.4706	0.4738	0.4305
7.00	0.8317	0.6588	0.4538	0.6118	0.5643	0.4505	0.4049	0.4009	0.3576
10.00	0.7789	0.5953	0.3979	0.5353	0.4808	0.3750	0.3356	0.3253	0.2848
15.00	0.7021	0.5125	0.3308	0.4412	0.3836	0.2918	0.2602	0.2460	0.2112
20.00	0.6364	0.4476	0.2817	0.3728	0.3168	0.2370	0.2108	0.1961	0.1664
25.00	0.5792	0.3950	0.2436	0.3205	0.2677	0.1979	0.1757	0.1617	0.1360
30.00	0.5289	0.3514	0.2132	0.2791	0.2300	0.1685	0.1494	0.1363	0.1141
35.00	0.4842	0.3145	0.1882	0.2455	0.2002	0.1456	0.1290	0.1169	0.0974
40.00	0.4443	0.2829	0.1674	0.2176	0.1759	0.1272	0.1126	0.1015	0.0842

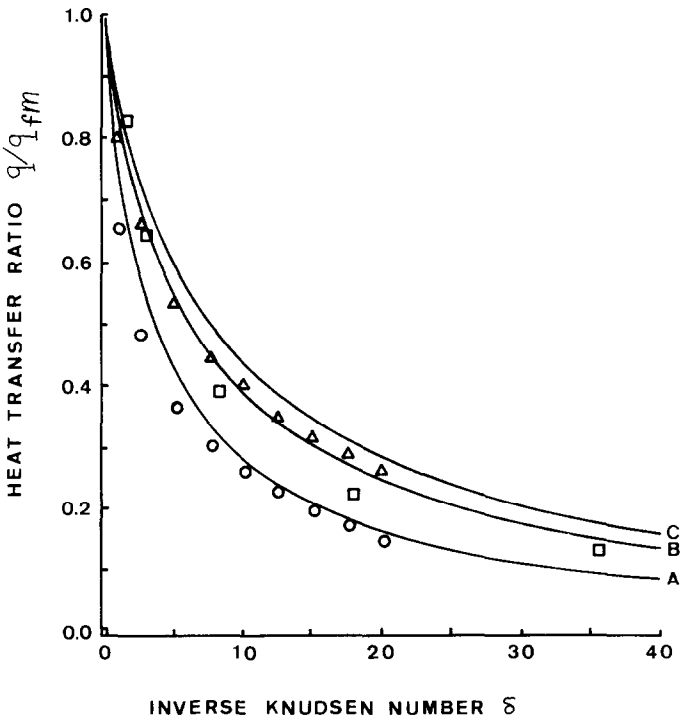

FIG. 1. Heat transfer ratio vs inverse Knudsen number. Present work ——; (A) $\alpha_{int} = 1$, $\alpha_{tr} = 1$, $Z = 5.94$, $F_t = 1.96$, $G = 1$. Hsu and Morse variational result, ○; (B) $\alpha_{int} = 0.5$, $\alpha_{tr} = 0.75$, $Z = 5.94$, $F_t = 1.96$, $G = 1$. Hsu and Morse variational result, □; (C) $\alpha_{int} = 0.76$, $\alpha_{tr} = 0.76$, $Z = 5.08$, $F_t = 1.96$, $G = 1$. Teagan and Springer experimental data, △.

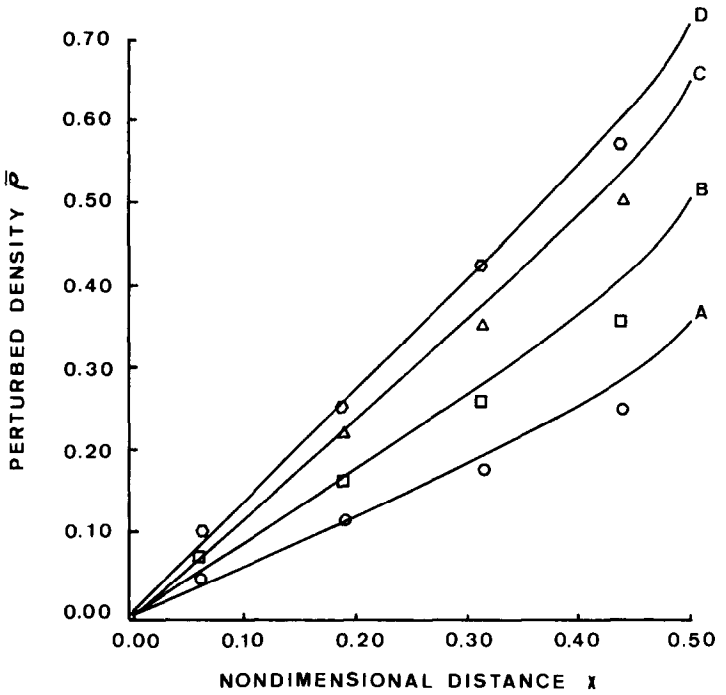

FIG. 2. Perturbed density profile for N_2 with $\alpha_{int} = 0.76$, $\alpha_{tr} = 0.76$, $Z = 5.08$, $F_t = 1.96$, $G = 1$. Present work ——; Teagan and Springer experimental data: (A) $\delta = 2.58$, ○; (B) $\delta = 5.6$, □; (C) $\delta = 11.25$, △; (D) $\delta = 15.5$, ○.

as δ increases. On the other hand, the experimental values exhibit more or less a straight line profile through the whole range. Unfortunately, the experimental data are too few and none near the plate to draw any firm conclusion from the comparison. The agreement, however, is quite good.

Table 5. Nitrogen temperature and density for inverse Knudsen number ($1/Kn$) of 2: $\alpha_{tr} = 1$, $\alpha_{int} = 1$, $F_t = 1.96$, $Z = 5.08$, $G = 1$

x	$\bar{\rho}$	$\bar{T}_{tr}$	$\bar{T}_{int}$	$\bar{T}_t$
−0.9989	−0.4475	0.7448	0.5094	0.6507
−0.9931	−0.4399	0.7323	0.5010	0.6398
−0.9815	−0.4279	0.7125	0.4877	0.6226
−0.9640	−0.4130	0.6876	0.4708	0.6009
−0.9408	−0.3960	0.6591	0.4516	0.5761
−0.9122	−0.3775	0.6278	0.4304	0.5489
−0.8783	−0.3576	0.5942	0.4076	0.5195
−0.8391	−0.3364	0.5585	0.3833	0.4884
−0.7950	−0.3143	0.5212	0.3579	0.4559
−0.7463	−0.2912	0.4824	0.3314	0.4220
−0.6932	−0.2674	0.4423	0.3041	0.3870
−0.6361	−0.2427	0.4012	0.2759	0.3511
−0.5751	−0.2175	0.3590	0.2470	0.3142
−0.5109	−0.1916	0.3160	0.2175	0.2766
−0.4436	−0.1652	0.2722	0.1875	0.2383
−0.3737	−0.1384	0.2278	0.1569	0.1995
−0.3016	−0.1112	0.1829	0.1260	0.1601
−0.2278	−0.0836	0.1375	0.0948	0.1204
−0.1526	−0.0559	0.0919	0.0633	0.0804
−0.0765	−0.0280	0.0460	0.0317	0.0403
0.0000	0.0000	0.0000	0.0000	0.0000

In conclusion our work has been effective in establishing the usefulness of the Hanson–Morse kinetic model for the problem of plane heat transfer in polyatomic gases. Future work will be directed toward extension to heat transfer in cylindrical and spherical geometries.

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APPENDIX

The expressions $S_m(x)$, $J_m(x)$, $L_m(x)$ and $K_{mj}(x, x')$ are given as follows:

$$S_m(x) = [(\varphi_m(\mathbf{c}, \varepsilon_i), 1)] \tag{A1}$$

$$J_m(x) = [(\varphi_m(\mathbf{c}, \varepsilon_i), (c^2-2))] \tag{A2}$$

$$L_m(x) = [(\varphi_m(\mathbf{c}, \varepsilon_i), (\varepsilon_i - G))] \tag{A3}$$

where

$$[(\varphi, \Psi)] = \sum \int \mathrm{d}\mathbf{c} \frac{\exp(-c^2-\varepsilon_i)}{Q_0 \pi^{3/2}} \varphi \Psi \{\eta(c_x) \times \exp[-(\delta/2-x)/c_x] - \eta(-c_x)\exp[-(\delta/2-x)/c_x]\}$$

$$K_{mj}(x, x') = \sum \int \mathrm{d}\mathbf{c}\eta(c_x) \frac{\exp[-c^2-\varepsilon_i-|x'-x|/c_x)]}{Q_0 \pi^{3/2}} \frac{1}{c_x} \times \psi_m[c_x \operatorname{sgn}(x-x'), c_y, c_z, \varepsilon_i]\psi_j[c_x \operatorname{sgn}(x-x'), c_y, c_z, \varepsilon_i]. \tag{A4}$$

After simplification, the above expressions are expressed in terms of Abramowitz functions:

$$T_n(x) = \int_0^\infty t^n \exp(-t^2 - x/t)\,\mathrm{d}t. \tag{A5}$$

The elements of matrix **C** in equation (37) are defined as follows:

$$C_{11} = 1 + 2T_1(\delta) + 2\int_{-\delta/2}^{\delta/2} \sum_{m=1}^{5} H_m(x)P_m(x)\,\mathrm{d}x \tag{A6}$$

$$C_{12} = 2[T_3(\delta) - T_1(\delta)] + 2\int_{-\delta/2}^{\delta/2} \sum_{m=1}^{5} H_m(x)Q_m(x)\,\mathrm{d}x \tag{A7}$$

$$C_{13} = 2\int_{-\delta/2}^{\delta/2} \sum_{m=1}^{5} H_m(x)R_m(x)\,\mathrm{d}x \tag{A8}$$

$$C_{21} = (1-\alpha_{tr})[T_3(\delta) - T_1(\delta)] + \int_{-\delta/2}^{\delta/2} \sum_{m=1}^{5} D_m(x)P_m(x)\,\mathrm{d}x \tag{A9}$$

$$C_{22} = 1 + (1-\alpha_{tr})[T_5(\delta) - 2T_3(\delta) + 2T_2(\delta)] \times \int_{-\delta/2}^{\delta/2} \sum_{m=1}^{5} D_m(x)Q_m(x)\,\mathrm{d}x \tag{A10}$$

$$C_{23} = (1-\alpha_{tr}) \int_{-\delta/2}^{\delta/2} \sum_{m=1}^{5} D_m(x)R_m(x)\,\mathrm{d}x \tag{A11}$$

$$C_{31} = (2/G)(1-\alpha_{int}) \int_{-\delta/2}^{\delta/2} \sum_{m=1}^{5} F_m(x)P_m(x)\,\mathrm{d}x \tag{A12}$$

$$C_{32} = (2/G)(1-\alpha_{int}) \int_{-\delta/2}^{\delta/2} \sum_{m=1}^{5} F_m(x)Q_m(x)\,\mathrm{d}x \tag{A13}$$

$$C_{33} = 1 + 2(1-\alpha_{int})T_1(\delta) + (2/G)(1-\alpha_{int}) \times \int_{-\delta/2}^{\delta/2} \sum_{m=1}^{5} F_m(x)R_m(x)\,\mathrm{d}x \tag{A14}$$

where

$$H_m = [1, \psi_m(\mathbf{c}, \varepsilon_j)] \tag{A15}$$

$$D_m = [1, (c^2-2)\psi_m(\mathbf{c}, \varepsilon_j)] \tag{A16}$$

$$F_m = [1, (\varepsilon_j - G)\psi_m(\mathbf{c}, \varepsilon_j)] \tag{A17}$$

and

$$(f, g) = \sum \frac{1}{\pi} \int_{\mathbf{c}\cdot\hat{n}<0} \mathrm{d}\mathbf{c} \frac{\exp[-c^2-\varepsilon_j+(\delta/2+x)/c_x]}{Q_0} [\mathbf{c}\cdot\hat{n}] fg. \tag{A18}$$

TRANSFERT THERMIQUE DANS UN GAZ POLYATOMIQUE RAREFIE—I. PLANS PARALLELES

Résumé—Le modèle de Hanson–Morse pour un gaz polyatomique est utilisé pour étudier le transfert thermique entre deux plans parallèles. Le problème de valeur limite est transformé en un système d'équations intégrales qui est résolu numériquement. On examine la dépendance des profils de transfert thermique, de température et de densité sur le nombre de Knudsen, l'énergie interne, le facteur d'Eucken et le nombre de relaxation de collision. Les résultats sont comparés à des données expérimentales sur l'azote et à des résultats variationnels publiés antérieurement sur le transfert thermique.

WÄRMEÜBERTRAGUNG IN EINEM VERDÜNNTEN MEHRATOMIGEN GAS—I. EBENE PARALLELE PLATTEN

Zusammenfassung—Das Hanson–Morse-Modell für ein mehratomiges Gas wird benutzt, um den Wärmeübergang zwischen zwei ebenen parallelen Platten zu untersuchen. Das Grenzwert-Problem wird in ein System von Integral-Gleichungen transformiert, das dann numerisch gelöst wird. Die Abhängigkeit des Wärmeübergangs, der Temperatur und des Dichteprofils von der Knudsen-Zahl, der inneren Energie, dem Eucken-Faktor und der Kollisions–Relaxations-Zahl wurde untersucht. Die Ergebnisse werden mit verfügbaren experimentellen Daten von Stickstoff und einigen früher veröffentlichten Ergebnissen des Wärmeübergangs verglichen.

ТЕПЛОПЕРЕНОС В РАЗРЕЖЕННОМ МНОГОАТОМНОМ ГАЗЕ—I. ПЛОСКОПАРАЛЛЕЛЬНЫЕ ПЛАСТИНЫ

Аннотация—Модель Хансона–Морзе для многоатомного газа применяется при изучении теплопереноса между двумя плоскопараллельными пластинами. Граничная задача сводится к системе интегральных уравнений, которые затем решаются численно. Исследуется зависимость теплопереноса, профилей температуры и плотности от числа Кнудсена, внутренней энергии, фактора Эйкена и параметра столкновительной релаксации. Полученные результаты сравниваются с имеющимися экспериментальными данными для азота и некоторыми ранее полученными вариационным методом результатами по теплопереносу.